ONLINE ISSN: 3007-3197 | PRINT ISSN: 3007-3189
http://amresearchreview.com/index.php/Journal/about

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 3, Issue 6 (2025)

Comparative Analysis of Finite Difference Schemes for the Two-Dimensional Heat Equation with Emphasis on the ADI Method

Sohail Ahmed Memon*[1], Israr Ahmed[2], Shabana Phulpoto[3], Mashooque Ali Mahar[4]

Article Details

Keywords:
Heat Equation, Finite Difference Method, FTCS, BTCS, Crank-Nicolson Method, Alternating Direction Implicit (ADI), Stability Analysis, Numerical Simulation, Partial Differential Equations, Computational Physics

[1]**Sohail Ahmed Memon (Corresponding Author)**
Department of Mathematics,
Shah Abdul Latif University, Khairpur Mirs
suhail.memon@salu.edu.pk
[2]**Israr Ahmed**
Department of Mathematics,
Shah Abdul Latif University, Khairpur Mirs
israr.memon@salu.edu.pk
[3]**Shabana Phulpoto**
Department of Mathematics,
Shah Abdul Latif University, Khairpur Mirs
hk0107144@gmail.com
[4]**Mashoque Ali Mahar**
Institute of Computer Science,
Shah Abdul Latif University, Khairpur Mirs
mashooq.mahar@salu.edu.pk

ABSTRACT

The heat equation, a partial differential equation in Mathematics and Physics, plays a key role in various fields including probability theory, Brownian motion, and financial mathematics. This work presents a comparative study of various finite difference schemes, comprised of Forward Time Central Space (FTCS), Backward Time Central Space (BTCS), Crank-Nicolson (CN), and the Alternating Direction Implicit (ADI) method for approximating the two-dimensional Heat equation. The FTCS method, despite its simplicity and computational efficiency, is known to be conditionally stable and has limitations for large-scale problems. Implicit methods such as BTCS and CN offer improved stability at the cost of increased computational complexity. The ADI method, designed to handle large sparse systems efficiently, emerges as a robust alternative for high-dimensional and computationally intensive problems. This work employs these schemes, evaluates their stability and computational performance, and compares the simulation results against existing benchmarks. The findings highlight the advantages and limitations of each approach, with a particular focus on the suitability of the ADI method for practical applications involving two-dimensional heat conduction.

1. Introduction

The heat equation is mathematical equation used to study the flow of heat conduction in a metal rod or in a metal square plate. The Heat equation is a popular equation that is regarded as the heat diffusion equation. The heat equation is solved analytically and as well as it is approximated numerically. The Heat equation analysis the physical phenomena that how does a hot body decreases in temperature according the to the atmosphere of the surroundings [1]. The heat equation has wider applications in probability and then financial engineering where the random walks are commonly studied. This equation serves as the model in Financial Mathematics. This equation is used computer algorithms based on the financial markets. The equation is used as model equation

DOI: Availability

for various applications. Variety of mathematical methods are tested using this equation. There are many other equations which deduce from this equation such the Black-Scholes equation [2], [3] used in financial mathematics is small variant of the heat equation. Further, many finite difference schemes are implemented using this model equation. The non-linear and complex partial differential equations implementation for discretization or computer simulations then this equation is used as a model equation [4]. Consideration of this equation in higher dimensions opens many new ways of investigations such as for two-dimensions the spatial part of the equation becomes a Laplacian equation [5].

Considering the widespread usage of the heat equation, the aim of this research is set to do work on the best finite difference scheme using the two-dimensional heat equation. In this study, the heat equation will be approximated using different finite difference schemes and then it will be discretized in Alternating Direction Implicit (ADI) scheme for the numerical approximations.

The ADI method was first introduced by Peaceman and Rachord [6]. Their work was regarded remarkable. Basically, the ADI method had essentially solved the problem of using complex and sparse matrices which are used in other finite difference schemes such as Crank-Nicolson method [7]. This method proved to be outperforming for the higher grids and the parallel computing problems.

There are some limitations of applying commonly used finite difference schemes for Heat equation which include Forward time, Central Space (FTCS) method, Backward time, Central Space (BTCS) method and Crank-Nicolson (CN) method.

The FTCS method which is well known as Forward Euler method, is considered unstable method though it is fast. Basically, The FTCS is an explicit scheme which is mainly considered limited for the finite difference schemes therefore implicit schemes are more preferred such Crank-Nicolson method. As the Crank-Nicolson method uses the huge and sparse matrices which require costly computations for bigger problems. The ADI scheme has been the motivation for such big and complex problems. We employ all finite difference to approximate the compare the results.

Except the Introduction section, other sections are included. The literature review is given in section 2, mathematical formulation of heat equation is presented in section 3, Finite difference methods and the numerical approximations are given in section 4. The conclusions are given in section 5.

2. Literature Review

The Heat equation is mainly used in the field of Physics where the magnetic fields release geothermal gases [8]. The finite difference scheme, with aided help of computers, based on the Heat equation to study on steady natural convection flow for effects of thermal radiation on a vertical plane surface [9]. Furthermore, the finite difference schemes were implemented for the mixed convection or radiation consequences of unsteady natural convection using the Heat equation [8], [10]. In a study, the phenomena of oscillatory flows was solved analytically using the Heat equation [11]. The Heat equation plays a key role in random movements, especially for the Brownian motion, this equation is used to investigate the random movement of bodies in a fluid flow. It has applications in financial mathematics where Brownian motion is used to understand the stock market patterns which become easy to predict the future sales. Another important equation, well properly known as the Black-Scholes equation, is modelled using the Heat equation model in the vast filed of financial mathematics. It is applied in diversified fields to approximate the partial differential equations of very complex problems and to solve scientific mysteries. It has also been used to solve Lie symmetry theory to decrease the overall symmetry in Newtonian fluids [8].

The finite difference methods come under the numerical analysis, a branch of mathematics. The differential equations are discretized on specified gird using finite difference methods to approximate numerically. The approximations are basically the transformations of derivatives in difference schemes [9]. The difference formulas are then analyzed based on their stability and convergence to obtain optimal accuracies. Thus, the differential equations for various systems portray the simulations.

Exploring the physical phenomena where the partial differential equations are used, a deep insight can be expressed. A variety of steady-state and time evolution problems are brought into the approximate solutions using finite difference methods but most of these problems are not solvable by analytical approach due to their initial and boundary conditions considerations. These physical phenomena involve mathematical based equations which involve physics which are continuous element dynamics, quantum mechanics, electrodynamics and furthermore, mass and heat-transfer theories are well described by the researchers [12], [13]. The mathematical methods for the solution of such equations of these physical phenomena involve the use of Riemann Integration, Green's Function and Fourier Method.

For the solution of some of these equations, they use methods such as the Fourier Method, the Riemann Integration, and the use of the Green's Function. Always these methods do not yield the exact analytical solution and this where the numerical techniques come into play and yield the approximate solution as close to the solution of the problem [14], [15], [16].

The Harriot [17], [18], [19], [20] developed numerical methods which were used to solve equations and later these developed methods were applied in engineering problems in 1943. In the same, finite element method, a numerical method was developed by Courant. The Courant on the other hand applied the variation method of Ritz to obtain the numerical solution for the mechanical systems [21], [22]. A necessary condition of convergence was named with Courant-Friedrichs-Lewy (CFL) used in obtaining the approximate solution of partial differential equations in partial differentials. The importance of this condition is known when the differential equations are carried out through the discrete schemes and the time step value is obtained to specific value (a characteristic value), if not taken care of the step value the code may be unstable and numerical values may diverge.

Thus, the importance of simulations can be recognized as the effective tool to approximate the solution and predict the convergence. The study of numerical methods for the simulations to do in an accurate way, requires a thorough study of the fundamentals of the numerical schemes with proper convergence criteria. The complexity of algorithms used to describe become easy after applying the specific discrete schemes used in the numerical methods. The model and the simulation process are two different things, at first the model is prepared which is regarded as the abstract of the steps described in the phenomena and the putting those steps into the practical is regarded as the simulation process. Usually, a mathematical model consists of the partial differential, initial values and the boundary conditions and it is brought into simulations with careful planning by using algorithms to solve the problem specifically. The numerical process come into existence when the mathematical problems based real phenomena become hard to determine the solution through the analytical methods. To approach such problems the simulations are used based on the algorithms which yield a contrasted solution of the problem and the results obtained from these simulations are compared hypothetical ideas or earlier anticipated solutions.

As in many cases the analytical solution is hardly possible, especially in the mathematical problems with regions of non-regular geometry. The finite difference schemes play a vital role for the numerical solution of such problems. However, the convergence of code and stability tests determine the best finite difference scheme to be used. Generally, the finite difference methods replace the continuous derivatives into discrete schemes and a fine mash or grid is used to fit discrete formulas, a mesh or grid is shown in the Figure 1 where the numerical solution is assumed to be solved.

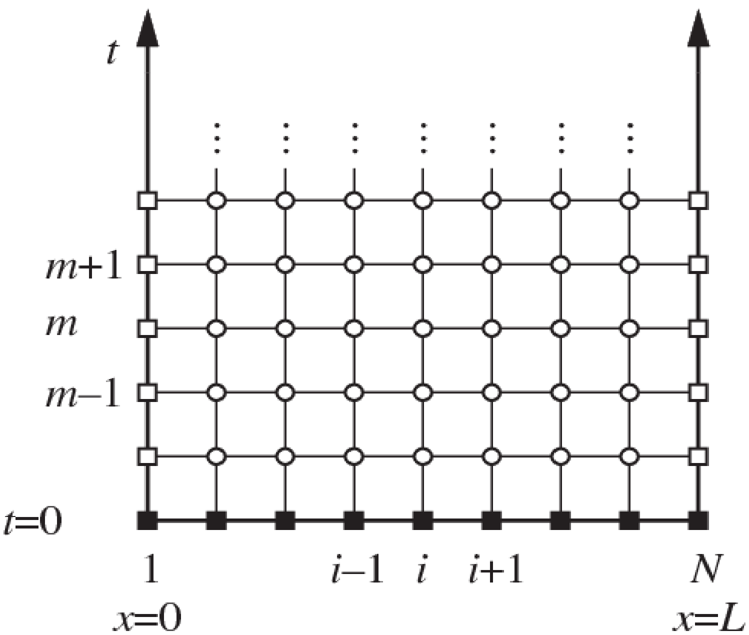

Figure 1. Mesh used for the solution of the Heat Equation [23].

3. Mathematical Formulation of the Heat Equation

As an example of the hot body is hot metal rod or a wire with a certain Length, say L. In such context, the mathematical equation forms as one-dimensional Heat equation, the one-dimensional Heat equation can be described as follows:

$u(x, y)$= temperature of the body
Then the 1-D form of the heat equation becomes:

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L \; and \; 0 \le t \le T, \tag{1}$$

where the constant K is for the thermal diffusivity, and is known as constant coefficient and u is an order parameter. The equation (1) is base version of the one-dimensional Heat equation. This can further be extended in two-dimensions, the phenomena can be assumed for the heat in a square plate of Length $L \times L$. The two-dimensional heat equation for the square plate can be represented as:

Let

$u(x, y, t)$ = temperature of the body,

The two-dimensional heat equation [24], is given as:

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),\ \ 0 \le x \le L, 0 \le y \le L \ and\ 0 \le t \le T, \tag{2}$$

where the constant coefficient K is the thermal diffusivity and the two-dimensional heat equation is subject to following initial condition:

$$u(x, y, t) = f(x, y) \ \ where\ (x, y) \in R, \tag{3}$$

And for the edges of the plate, the Dirichlet boundary conditions are given as:

$$\begin{aligned} u(0, y, t) = u(L, y, t) &= 0 \\ u(x, 0, t) = u(x, L, t) &= 0 \end{aligned} \tag{4}$$

a. Analytical solution of one-dimensional Heat equation

In this section, the analytical solution of one-dimensional heat equation is briefly described. The analytical solution can be carried using separation of variables technique in three steps. Consider the heat flow in a uniform rod of length L and $x \in [0, L]$ using equation (1) with following boundary and initial conditions:

Boundary conditions: (5)

$u(0, t) = 0$ for $t \ge 0$,

$u(L, t) = 0$ for $t \ge 0$.

Initial condition: (6)

$u(x, 0) = f(x)$ for $x \in [0, L]$

After applying initial condition given in equation (6) in analytical solution of equation (1), the general solution:

$$u(x, t) = \sum_{n}^{\infty} u_n(x, t) \tag{7}$$

becomes:

$$u(x, t) = \sum_{n}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right) e^{-\frac{Kn^2\pi^2}{L^2}t} \tag{8}$$

The equation (8) is the analytical solution of the one-dimensional heat equation (1). The B_n in equation (8) is used as the Fourier coefficient.

Let $f(x) = 100\sin(\pi x/80)$, and $K = 1,\ L = 80cm$ for the following equation:

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = f(x) \tag{9}$$

By comparing coefficients, we obtain:

$$u_n(x,t) = 100\sin\left(\frac{\pi}{80}x\right)\, e^{-\frac{K^2\pi^2}{(80)^2}t}. \tag{10}$$

The solution was obtained using boundary conditions and applying initial condition. The obtained solution passed through various mathematical procedures which included ordinary differential equations and Fourier series. The plot of the analytical solution based on the chosen function is shown in Figure 2.

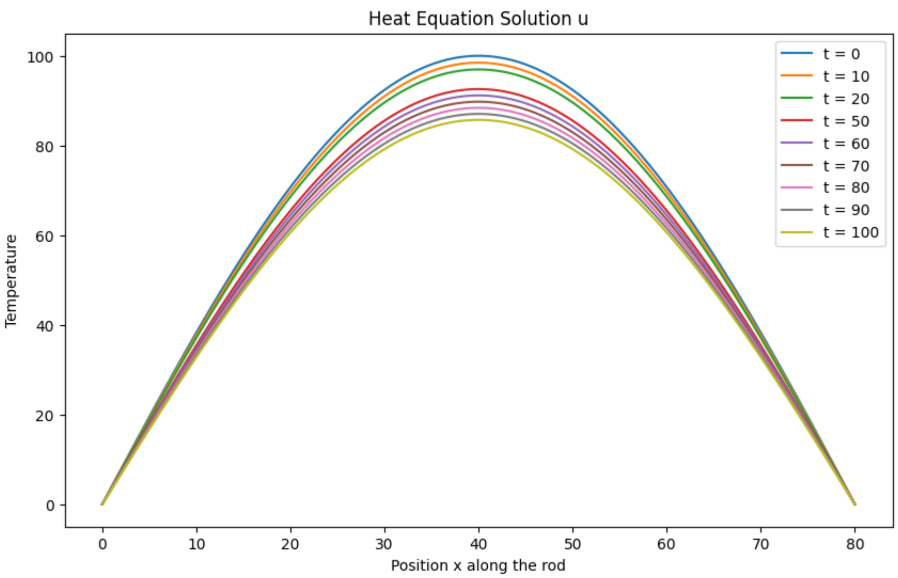

Figure 2. Plotting of the analytical solution of the Heat equation (1)

4. Finite Difference Methods for the Heat Equation

In this section, the approximate solution of the two-dimensional Heat equation has been presented using various finite difference schemes. Here, mainly four finite difference schemes are used for the two-dimensional Heat equation (2) which comprise the Forward Euler method or Forward in Time, Centered in Space (FTCS), Backward Euler method or Backward in Time, Centered in Space, the Crank-Nicolson scheme and the Alternating Direction Implicit Method (ADI). These all schemes are presented with their stability analysis.

The two-dimensional heat equation (2) can be discretised in FTCS as follows:

$$\frac{\partial u}{\partial t} = \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} \tag{11}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{(\Delta x)^2} \tag{12}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{(\Delta y)^2} \tag{13}$$

a. Forward Time and Centered Space (FTCS)

The FTCS, an explicit finite difference scheme, is also called the Forward Euler method and approximates the numerically in forward space. The substitutions of equations (11), (12) and (13) in equation (2), the two-dimensional Heat equation becomes [3]:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = K\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{(\Delta x)^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{(\Delta y)^2}\right) \tag{14}$$

After the algebraic manipulation and setting the values $r_x = K\frac{\Delta t}{(\Delta x)^2}$ and $r_y = K\frac{\Delta t}{(\Delta y)^2}$, the equation (14) becomes:

$$u_{i,j}^{n+1} = u_{i,j}^n(1 - 2r_x - 2r_y) + r_x\left(u_{i+1,j}^n + u_{i-1,j}^n\right) + r_y(u_{i,j+1}^n + u_{i,j-1}^n) \tag{15}$$

The equation (15) is the discretised in FTCS finite difference scheme that approximates the two-dimensional equation (2).

The stability analysis of equation (14) ensures the approximate value of time step Δt and can be represented using Fourier term [3] as:

$$u_{i,j}^n = \xi^n e^{iip\pi\Delta x} e^{ijq\pi\Delta x} \tag{16}$$

The substitution of $u_{i,j}^n$ from equation (16) into equation (15), and the resulting equation divided by ξ^{n+1} becomes:

$$\xi = 1 - 4r_x \sin^2\left(\frac{p\pi\Delta x}{2}\right) - 4r_y \sin^2\left(\frac{q\pi\Delta y}{2}\right) \tag{17}$$

Which results in the Neumann criterion $|\xi| \leq 1$ where $r_x + r_y \leq \frac{1}{2}$ and this result limits Δt to be chosen smaller. The FTCS has truncation error of $O(\Delta t) + O(\Delta x^2)$ which is first-order in time and second-order in space [3].

A 3D surface plot has been shown in Figure 3 from the numerical approximation of two-dimensional Heat equation (2) based on the FTCS method (15), the configurations are $f(x) = 100\sin(\pi x/80)$, and $K = 1$, $L = 80cm$, which have been used for a graph in Figure 2 for analytical solution of one-dimensional Heat equation (1).

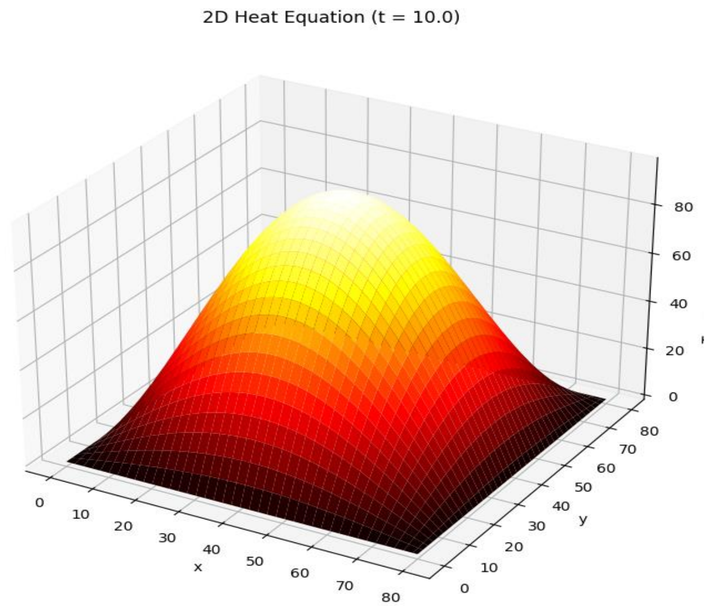

Figure 3. Surface plot of 2D Heat equation using FTCS method.

b. Backward Time Centred Space (BTCS)

The BTCS finite difference is an implicit Backward Euler method where approximation take place on left hand side in the time space from step space on right hand side, a behaviour that is opposite to the explicit FTCS. Same substitution of equations (11), (12) and (13) in equation (2) is done but with slight change in equation (12) and (13), the value of n is set to $n+1$. The equation of BTCS is then obtained:

$$(1 + 2r_x + 2r_y)u_{i,j}^{n+1} - r_x\left(u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1}\right) - r_y\left(u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}\right) = u_{i,j}^{n} \qquad (18)$$

The equation (18) works in way that the values of u^{n+1} are calculated by the help a Matrix which uses the technique of system of equations solving. Say, $Ax = b$ form where A is the matrix of the size $(N_x - 1) \times (N_x - 1)$ and b is a vector matrix. There are several iterative methods to solve for example LU decomposition method, Crout factorization method and Conjugate gradient method, etc. Finally, the system of equations becomes inverse matrix $x = A^{-1}b$ form to be solved.

The substitution of equation (16) with a slight change of value $n = n + 1$ in equation (18) and dividing by ξ^{n+1}, the stability equation takes the form:

$$\xi = \frac{1}{1 + 4r_x sin^2\left(\frac{p\pi\Delta x}{2}\right) + 4r_y \sin^2\left(\frac{q\pi\Delta y}{2}\right)} \qquad (19)$$

The equation (19) ensures the stability for the time step Δt for BTCS scheme where $|\xi| \leq 1$. The limitation $r \leq 1/2$ becomes unnecessary. Thus, BTCS becomes unconditionally stable by making time step Δt independent to be used for any value. The error order of implicit BTCS scheme and explicit FTCS scheme has same, the first-order in time and second-order in step space.

A 3D surface plot has been shown in Figure 4 from the numerical approximation of two-dimensional Heat

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 3, Issue 6 (2025)

equation (2) based on BTCS method using same configuration that of FTCS.

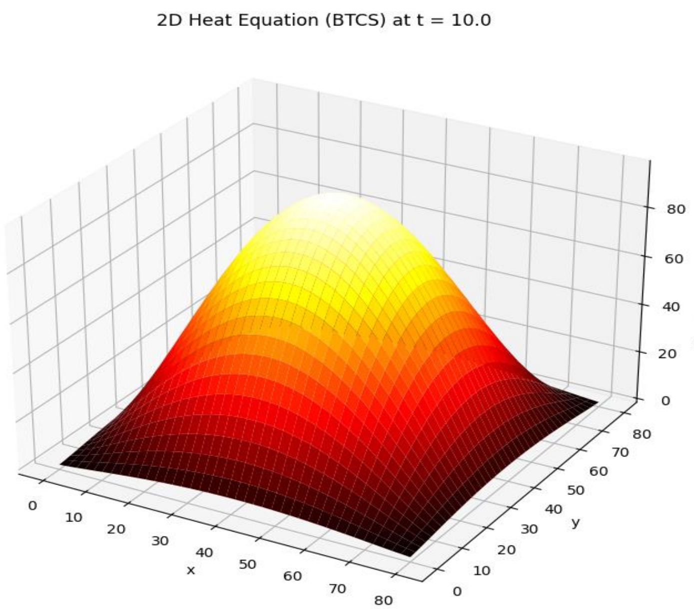

Figure 4. Surface plot of 2D Heat equation using BTCS method.

c. Crank-Nicolson (CN)

The CN scheme was first proposed by Crank & Nicolson [25]. This scheme was adopted in finite difference methods to make the methods more stable and unconditional for Δt. The CN has been employed in finite difference schemes to approximate the differential equations for various applications such as Financial Mathematics, Physics, and Engineering. It is considered a robust implicit finite difference scheme than BTCS and FTCS. The CN scheme results in second-order truncation error in time and in step. Despite, the CN scheme being more stable and second-order accurate, it produces uncontrolled oscillations in numerical calculations. The oscillations are caused by using matrices with bigger size in complex problems for non-linear partial differential equations [26].

The formation of CN scheme is based on averaging the explicit FTCS and implicit BTCS schemes by using the central difference derivatives. After the substitution of FTCS and BTCS equations (15) and (18) in equation (2), the CN scheme becomes:

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t} = K\left(\frac{\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{\Delta x^2}+\frac{u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}}{\Delta y^2}+\frac{u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{\Delta x^2}+\frac{u_{i,j+1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{\Delta y^2}}{2}\right) \tag{20}$$

The equation (20) is the discretised CN method of two-dimensional equation (2). The variable u, in this scheme, is approximated by an average of n and $n+1$ spaces.

Further, simplifying and setting the terms, given as follows:

Let, $\Delta_2^{(x)} u_{i,j}^n = u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n$ and $\Delta_2^{(y)} u_{i,j}^n = u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j+1}^n$ and likewise setting for $\Delta_2^{(x)} u_{i,j}^{n+1}$ and $\Delta_2^{(y)} u_{i,j}^{n+1}$. Then, substituting these terms in equation (20) along with other terms where $r_x = K\frac{\Delta t}{\Delta x^2}$ and $r_y = K\frac{\Delta t}{\Delta y^2}$, the CN equation (2) becomes:

$$(1 - \frac{r_x}{2}\Delta_2^{(x)} - \frac{r_y}{2}\Delta_2^{(y)})u_{i,j}^{n+1} = (1 + \frac{r_x}{2}\Delta_2^{(x)} + \frac{r_y}{2}\Delta_2^{(y)})u_{i,j}^n \tag{21}$$

For numerical approximation, the equation (21) can further be simplified by introducing the matrices such as:

$$A_1 u_{i,j}^{n+1} = A_2 u_{i,j}^n \tag{22}$$

The A_1 and A_2 are two-dimensional matrices of size $(N_x - 1)(N_y - 1) \times (N_x - 1)(N_y - 1)$. Such system of equations can be solved by involving Conjugate Gradient method or LU decomposition method to approximate $u_{i,j}^{n+1}$.

The stability of implicit CN method for two-dimensional Heat equation is derived using discrete Neumann criterion involving Fourier terms. The stability analysis follows from the FTCS and BTCS, the amplification factor for CN scheme (21) in terms of stability is give as:

$$\xi = \frac{1 - 2r_x \sin^2\left(\frac{p\pi\Delta x}{2}\right) - 2r_y \sin^2\left(\frac{p\pi\Delta x}{2}\right)}{1 + 2r_x \sin^2\left(\frac{p\pi\Delta x}{2}\right) + 2r_y \sin^2\left(\frac{p\pi\Delta x}{2}\right)}. \tag{23}$$

Assuming, $r_x \geq 0$ and $r_y \geq 0$ and $|\xi| \leq 1$ imply the CN method is unconditionally stable with truncation error of second-order accuracy.

A 3D surface plot has been shown in Figure 5 from the numerical approximation of two-dimensional Heat equation (2) based on CN method using same configuration that of BTCS and FTCS.

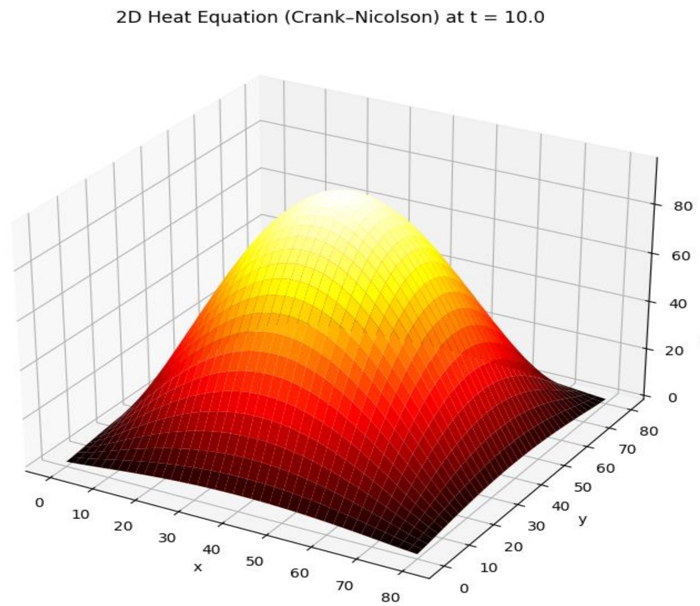

Figure 5. Surface plot of 2D Heat equation using CN method.

d. Alternating Direction Implicit Method (ADI)

The Alternating Direction Implicit method uses the operator splitting technique which was first proposed by Peaceman and Rachford [6], [27]. The ADI finite difference method is a stronger version of CN with speed and accuracy. The methodology of ADI is designed to avoid using large matrices and therefore wins over the CN and BTCS which involve large matrices for approximating the partial differential equations. The huge matrices consume memory at large extent and process very slowly, this happens specially in CN method. The ADI method works fast and is mostly employed for parallel processing, due to its nature of splitting operator [28].

The discretized equations of ADI for two-dimensional Heat equation (2) in the operator splitting manner are given as follows:

$$u_{i,j}^{n+\frac{1}{2}} - \frac{r_x}{2}\left(u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}\right) = u_{i,j}^{n} + \frac{r_y}{2}\left(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}\right) \quad (24)$$

$$u_{i,j}^{n+1} - \frac{r_y}{2}\left(u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}\right) = u_{i,j}^{n+\frac{1}{2}} + \frac{r_x}{2}\left(u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}\right) \quad (25)$$

The discretized equations (24) and (25) have been simplified further as follows:

$$\left(1 - \frac{r_x}{2}\Delta_2^{(x)}\right) u_{i,j}^{n+\frac{1}{2}} = \left(1 + \frac{r_y}{2}\Delta_2^{(y)}\right) u_{i,j}^{n} \quad (26)$$

$$\left(1 - \frac{r_y}{2}\Delta_2^{(y)}\right) u_{i,j}^{n+1} = \left(1 + \frac{r_x}{2}\Delta_2^{(x)}\right) u_{i,j}^{n+\frac{1}{2}} \quad (27)$$

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 3, Issue 6 (2025)

The operators $\Delta_2^{(x)}$ and $\Delta_2^{(y)}$ in equations (26) and (27) are for step i (x-coordinate) and step j (y-coordinate). The system of equations (simultaneous equations) (26) and (27) refer to the operator splitting method. Equation (26) approximates half-spaced derivative in step i as an implicit way (left-side) and step j in explicit way (right-side). Equation (27) approximates half-spaced derivative in step i as an explicit (right-side) way and step j in implicit way (left-side). The matrices are used to do numerical calculations similar to CN method in equation (22). Such methodology makes ADI approximation very fast and stable compared to the traditional finite difference methods.

For the stability analysis of ADI, the equations (26) and (27) are combined into one equation by the substitution of the value of $u_{i,j}^{n+\frac{1}{2}}$ from equation (26) into equation (27). The resulting equation can be seen as follows:

$$\left(1-\frac{r_x}{2}\Delta_2^{(x)}\right)\left(1-\frac{r_y}{2}\Delta_2^{(y)}\right)\boldsymbol{u}_{\boldsymbol{i,j}}^{\boldsymbol{n+1}}=\left(1+\frac{r_x}{2}\Delta_2^{(x)}\right)\left(1+\frac{r_y}{2}\Delta_2^{(y)}\right)u_{i,j}^{n} \qquad (28)$$

The substitution of the Fourier term given in equation (16) in equation (28) by using discrete stability (Von Neumann) analysis, the stability analysis equation for the ADI method for two-dimensional Heat equation is follows:

$$\xi=\frac{\left(1-2r_x\sin^2\left(\frac{p\pi\Delta x}{2}\right)\right)\left(1-2r_y\sin^2\left(\frac{p\pi\Delta x}{2}\right)\right)}{\left(1+2r_x\sin^2\left(\frac{p\pi\Delta x}{2}\right)\right)\left(1+2r_y\sin^2\left(\frac{p\pi\Delta x}{2}\right)\right)} \qquad (29)$$

Assuming the non-negative r_x and r_y and $|\xi|\le 1$ makes the ADI method unconditionally stable for Δt value and makes ADI method convergent with second-order accuracy.

A 3D surface plot has been shown in Figure 6 from the numerical approximation of two-dimensional Heat equation (2) based on ADI method using same configuration that of CN.

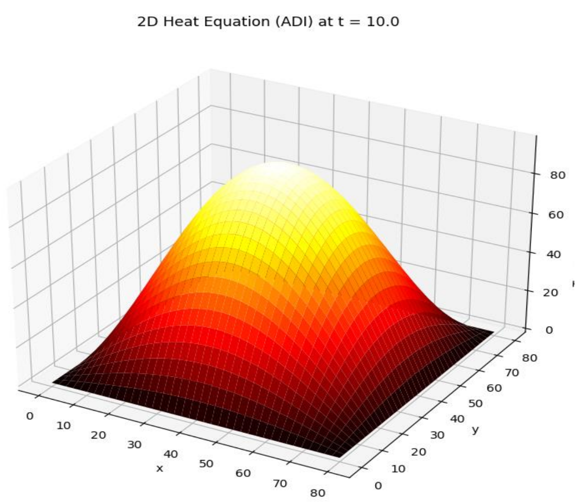

Figure 6. Surface plot of 2D Heat equation using ADI method.

5. Conclusions

The heat equation, which is also well popularly known as Heat Diffusion equation, has many applications in various fields such as Physics, Finance, Engineering, Astronomy, etc. For the purpose of simulation in computer code, the differential equations are discretized using finite difference methods for the numerical approximations.

In this work, the two-dimensional Heat equation (2) has been discretized using four different finite difference methods. The one-dimensional equation has been solved for heated-rod analytically, the examples of two-dimensional equation has been shown to be approximated for a square plate of size $L \times L$. The finite difference schemes used in this study include explicit Forward Euler (FTCS), implicit Backward Euler (BTCS), Crank Nicolson (CN) and Alternating Direction Implicit (ADI) method. The FTCS method approximates the two-dimensional equation in an explicit way. The discretization of two-dimensional using FTCS method has some limitations that Δt must be chosen smaller, it cannot be chosen 1 or greater than 1 for numerical approximations. It is first- order accurate in time second-order accurate in space. The BTCS method of discretization for the two-dimensional Heat equation has been shown to be unconditionally stable, the stability analysis for the BTCS was discussed. The BTCS method is an implicit method. The CN method has been discretized, the stability analysis for this scheme shows that the CN scheme is unconditionally stable. The CN scheme is an average of FTCS and BTCS and therefore it is more stable but slow in numerical approximation due to the use of large matrices for more complex problems. The CN method is second-order accurate in time and space. It must be noted that the BTCS and CN methods carry the system of equations which are solved using the LU decomposition or Conjugate Gradient method. The ADI method is stronger version of CN method and the its discretization scheme is based alternating direction of implicit and explicit using half-spaced derivative. This ADI method is fast in comparison of CN method and is unconditionally stable like CN method. The ADI method can be used for more complex problems and the numerical approximations can be obtained very fast. This method can also be carried out for the parallel computing. For each finite difference scheme used in this study, a 3D surface plot has been shown. These are generate using different approaches FTCS, BTCS, CN and ADI, the plots seem alike but are based on different techniques. There are various finite difference schemes for discretization of differential equations. The choice of these schemes can best be shown fit for parabolic, elliptical or hyperbolic differential equations based on the stability and convergence. Thus, the stable and time-efficient numerical results can be obtained.

6. References

[1] W. Gerald, "Finite–Difference Approximations to the Heat Equation," *Portland State Univ. Portland Or.*, 2011.
[2] R. L. Burden and J. D. Faires, "Numerical analysis. 1993," *PWS Boston*.
[3] D. J. Duffy, *Finite Difference methods in financial engineering: a Partial Differential Equation approach*. John Wiley & Sons, 2013.
[4] E. Bänsch and K. Mikula, "A coarsening finite element strategy in image selective smoothing," *Comput. Vis. Sci.*, vol. 1, no. 1, pp. 53–61, 1997.
[5] A. Kumar, "Isotropic finite-differences," *J. Comput. Phys.*, vol. 201, no. 1, pp. 109–118, 2004.
[6] D. W. Peaceman and H. H. Rachford, "The numerical solution of parabolic and elliptic differential equations," *J. Soc. Ind. Appl. Math.*, vol. 3, no. 1, pp. 28–41, 1955.
[7] J. Crank and P. Nicolson, "A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type," *Adv. Comput. Math.*, vol. 6, no. 1, pp. 207–226, 1996.
[8] M. A. El-Hakiem, "MHD oscillatory flow on free convection–radiation through a porous medium with constant suction velocity," *J. Magn. Magn. Mater.*, vol. 220, no. 2–3, pp. 271–276, 2000.

[9] N. Parekh and S. Puri, “A new numerical scheme for the Fisher equation,” *J. Phys. Math. Gen.*, vol. 23, no. 21, p. L1085, 1990.
[10] L. Alvarez, F. Guichard, P.-L. Lions, and J.-M. Morel, “Axioms and fundamental equations of image processing,” *Arch. Ration. Mech. Anal.*, vol. 123, no. 3, pp. 199–257, 1993.
[11] A. J. Chamkha and A.-R. A. Khaled, “Similarity solutions for hydromagnetic simultaneous heat and mass transfer by natural convection from an inclined plate with internal heat generation or absorption,” *Heat Mass Transf.*, vol. 37, no. 2, pp. 117–123, 2001.
[12] W. F. Ames, *Numerical methods for partial differential equations*. Academic press, 2014.
[13] L. Olsen-Kettle, “Numerical solution of partial differential equations,” *Lect. Notes Univ. Qld. Aust.*, 2011.
[14] S. C. Chapra, R. P. Canale, R. S. G. Ruiz, V. H. I. Mercado, E. M. Díaz, and G. E. Benites, *Métodos numéricos para ingenieros*, vol. 5. McGraw-Hill, 2011.
[15] J. M. Cooper, *Introduction to partial differential equations with MATLAB*. Springer Science & Business Media, 2012.
[16] K. W. Morton and D. F. Mayers, *Numerical solution of partial differential equations: an introduction*. Cambridge university press, 2005.
[17] A. Clive and A. Fletcher, “Computational Techniques for Fluid Dynamics,” *Ëpringer-Verl. Berl. Heidelb.*, 1988.
[18] M. N. Khan, I. Ahmad, A. Akgül, H. Ahmad, and P. Thounthong, “Numerical solution of time-fractional coupled Korteweg–de Vries and Klein–Gordon equations by local meshless method,” *Pramana*, vol. 95, no. 1, pp. 1–13, 2021.
[19] W. Liu *et al.*, “Ssd: Single shot multibox detector,” in *European conference on computer vision*, Springer, 2016, pp. 21–37.
[20] G. W. Recktenwald, “Finite-difference approximations to the heat equation,” *Mech. Eng.*, vol. 10, no. 01, 2004.
[21] G. H. Golub and J. M. Ortega, *Scientific computing: an introduction with parallel computing*. Elsevier, 2014.
[22] J. D. Hoffman and S. Frankel, *Numerical methods for engineers and scientists*. CRC press, 2018.
[23] Recktenwald GW, “Finite-difference approximations to the heat equation,” *Mech. Enginnering*, vol. 10, no. 01, Jan. 2004.
[24] M. N. Borjini, C. Mbow, and M. Daguenet, “Numerical analysis of the effect of radiation on laminar steady natural convection in a two-dimensional participating medium between two horizontal confocal elliptical cylinders,” *Numer. Heat Transf. Part Appl.*, vol. 35, no. 5, pp. 467–494, 1999.
[25] J. Crank and P. Nicolson, “A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type,” *Adv. Comput. Math.*, vol. 6, no. 1, pp. 207–226, 1996.
[26] P. Wilmott, S. Howison, and J. Dewynne, *The mathematics of financial derivatives: a student introduction*. Cambridge university press, 1995. Accessed: Jun. 10, 2025. [Online]. Available: https://books.google.com/books?hl=en&lr=&id=VYVhnC3fIVEC&oi=fnd&pg=PR9&dq=Wilmott+et+al.,+1995&ots=-R6jcVXav6&sig=v91qevHYUJgOrrnVxBcvbit3YMc
[27] S. Sirca and M. Horvat, *Computational methods for physicists: compendium for students*. Springer Science & Business Media, 2012. Accessed: Jun. 10, 2025. [Online]. Available: https://books.google.com/books?hl=en&lr=&id=hIPip3UxX3gC&oi=fnd&pg=PR4&dq=Sirca+and+Horvat+2012&ots=wgejeUBGVF&sig=__9jcAyDARFIBvjOATrp1AZ01JI
[28] I. J. Craig and A. D. Sneyd, “An alternating-direction implicit scheme for parabolic equations with mixed derivatives,” *Comput. Math. Appl.*, vol. 16, no. 4, pp. 341–350, 1988.