Online ISSN 3007-3197
Print ISSN 3007-3189
http://amresearchreview.com/index.php/Journal/about

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 3, Issue 4 (2025)

Numerical Analysis of Navier–Stokes Equations for Multi-Layer Incompressible Flows Utilizing Finite Difference Method and Finite Volume Method

Israr Ahmed[1], Faraz Ahmed Soomro[2], Sohail Ahmed Memon[3], Paras Soomro[4]

Article Details

Keywords: Stratified fluid flow; Multi-fluid oil transport; Finite Volume Method (FVM); Finite Difference Method (FDM); Navier–Stokes equations; Interface dynamics; Density stratification; Numerical simulation; Incompressible fluids; Octave.

Israr Ahmed
Associate Professor, Department of Mathematics Faculty of Physical Sciences, Shah Abdul Latif University, Khairpur Mir's
israr.memon@salu.edu.pk

Faraz Ahmed Soomro
Department of Mathematics,
Shah Abdul Latif University
Khairpur Mir's.
farazahmedsoomro01@gmail.com

Sohail Ahmed Memon
Associate Professor, Department of Mathematics Faculty of Physical Sciences, Shah Abdul Latif University, Khairpur Mir's
suhail.memon@salu.edu.pk

Paras Soomro
Department of Mathematics,
The Begum Nusrat Bhutto Women University Sukkur.
soomroparas858@gmail.com

ABSTRACT

This research focuses on the numerical simulation of stratified incompressible multi-fluid flows, particularly in multi-fluid oil transport systems. The traditional models built on homogeneous fluids do not capture well the complexities brought about by changes in density and viscosity across immiscible layers. To overcome this challenge, the Finite Difference Method (FDM) and Finite Volume Method (FVM) are applied to solve the Navier–Stokes equations in a layered manner. The computational domain is a two-dimensional pipeline geometry with an oil-water interface, undergoing isothermal and incompressible flow. The dynamics of the interface are described by stepwise density functions with continuity conditions for velocity and shear stress. The simulations are done in Octave, and results confirm the advantages of FVM in conservation properties, interface handling, and complex geometry applications. The results show that robust discretization techniques are critical for accurately predicting flow behavior in industrial multi-fluid transport systems.

Introduction

In developing numerical solutions to the Navier–Stokes equations, the focus has been on the easier cases of homogeneous incompressible fluids with constant density [1,2]. While the theoretical models development covers quite a range of hydrodynamic behavior in many time and length scales, the original models will fall short when faced with situations that are more complex. Specifically, in many engineering applications today, such as the multi-fluid transport of oil, we need to study a much more complex interaction of stratified fluid dynamics and more variable density systems. The FDM method and the FVM method will be used to build and investigate complex mathematical models numerically. By using Octave to simulate models numerically, many important details about layered flow and characteristics of the interfaces are gained and the value of these details are often critical for design, performance, and understanding of an industrial fluids transport system.

Multilayered structures commonly arise in isothermal flows of viscous incompressible fluids because of the density stratifications. When the fluid is incompressible, the density is a Varangian invariant [3, 4, 5] thus, properties of the velocity field can vary in moments and spatially. In a stratified fluid environment, the major change in flow behavior, by definition of isothermal flow, occurs in the vertical (transverse) direction. Vertical density stratification has significant implications for large structure dynamics, energy transfer between vertical structures, and the influencing of internal wave generation [6-10], however, horizontal (longitudinal) density gradients can affect different types of convective phenomena [11-14].

While stratification is, in principle, determined by continuous temporal and spatial functions, in practice it often appears that the density distribution may need to be approximated and perhaps might only be estimated. This limitation complicates matters and may lead to ill-posed problems in the general cases of hydrodynamics and mathematical physics. For these reasons, it is very common for researchers to utilize layered models that utilize a step function for density and for each layer assign values of density, and more usefully, dynamic viscosity. This simplification allows us to analyze large-scale flow instability and provides useful insights into the fluid behavior of stratified systems. The multilayer approximation, specifically 2-layer models, has been particularly useful in explaining important aspects of fluid behavior [15, 16, 17]. These models are also useful in simulating equatorial flow behavior as well as other naturally stratified systems [18, 19, 20, 21].

In the context of multi-fluid oil transport, these multilayer models provide a realistic yet computationally efficient representation of immiscible oil components traveling through pipelines or processing channels. The modeling of such systems often relies on the numerical solution of Navier–Stokes equations, with a focus on developing robust discretization techniques like the Finite Difference Method (FDM) and Finite Volume Method (FVM) [22, 23]. These methods enable accurate simulation of interfacial instabilities and mixing effects prevalent in optimizing industrial processes of complex fluid interactions.

With the complexity introduced from stepwise density distributions and interfacial dynamics governing stratified flows, particularly in multi-fluid systems, it is increasingly relevant to formulate computationally reliable and physically consistent solutions to the Navier–Stokes equations. Rather than targeting exact analytic solutions (which restrict the form and applicability of solutions), we want to develop numerically accurate solutions for stratified fluid flows constructed using step density functions. More specifically, we intend to evaluate and compare the numerical accuracy of the Finite Difference Method (FDM) and Finite Volume Method (FVM) and their application to modeling multi-fluid oil transport scenarios wherein layers of immiscible fluids are flowing at steady state, isothermal, incompressible flow.

Problem statement

The transport of immiscible, stratified fluids in pipes has many constraints with mathematical modeling in multiphase oil transport systems due to density, [24, 25] viscosity and interfacial complexities. This is important since an accurate characterization of this class of flow is necessary to forecast operational response, and avoid instability, blocking or energy loss in laterally lengthy systems.

This research is based on a reliable model for stratified incompressible viscous fluid flow, which is governed by Navier–Stokes equations. For each fluid layer i, the governing equations for the system are:

$$\rho_i\left(\frac{\partial \vec{v}^i}{\partial t}+\left(\vec{v}^i.\nabla\right)\vec{v}^i\right)=-\nabla P^i+\eta_i\nabla^2\vec{v}^i+\vec{F}^i, \qquad \nabla.\vec{v}^i=0$$

In this paper, we keep the original mathematical model, but apply two updated numerical models Finite Difference Method (FDM) and Finite volume method (FVM)) to solve the flow equations. These models are incredibly flexible when dealing with layer wise heterogeneity, boundary conditions, [26-28] and non-ideal geometries typical of real-world oil pipeline operations. All of the numerical simulations are completed using Octave.

Figure 1 shows a schematic of the physical domain as a horizontal pipeline in several layers of fluid. Each layer has a distinct density [29], viscosity ηi, and thickness. The arrows on the figure show the direction of flow (horizontally) and the effects of gravity g (vertically), as gravity is one of the forces acting on the system. It is very common for engineering pictures to not explicitly show the axis labels and coordinate axes, as the image is representing a more generalized engineering view of the flow and not detailed axes.

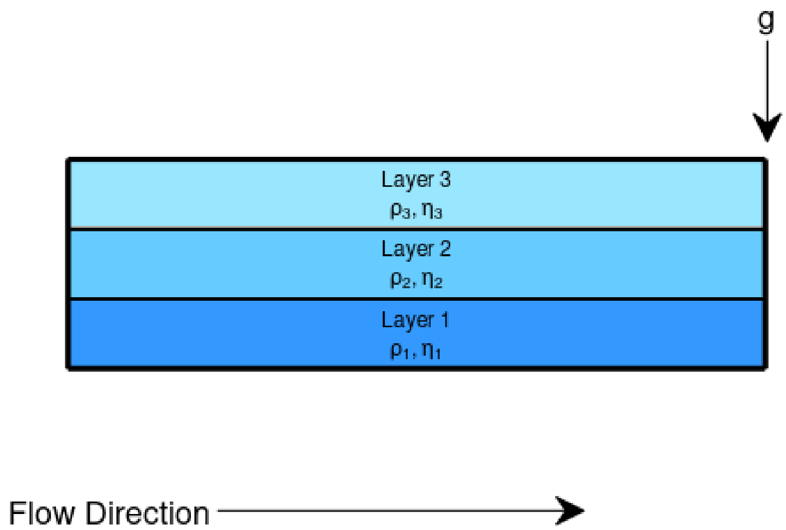

Figure 1: Schematic flow of a stratified fluid

This study's purpose is to assess the performance and precision of the Finite Difference Method (FDM) and the Finite Volume Method (FVM) applied to situations of this layered flow type with particular regard to their suitability to multi-fluid oil transportation problem contexts.

Domain and Fluid Description

The computational domain comprises a two-dimensional rectangular pipe with a length of L=1 meters and a height of H=0.2 meters. The domain is divided in half, with water (more viscous and dense) occupying the lower half and oil placed in the upper half.

Boundary Conditions:

- Walls: No-slip
- Inlet: Prescribed velocity profile
- Outlet: Constant pressure

Finite Volume Method (FVM)

The momentum equation is applied to a control volume V.

$$\frac{d}{dt}\int_V \rho\vec{v}\,dV + \int_{\partial V}\rho\vec{v}(\vec{v}.\vec{n})dA = -\int_{\partial V}P\vec{n}\,dA + \int_{\partial V}\eta\nabla\vec{v}\vec{n}\,dA + \int_V \vec{F}\,dV$$

The process of discretizing terms

- **Transient Term:**

$$\rho p\Delta x\Delta y\left(\frac{u_p^{n+1} - u_p^n}{\Delta t}\right)$$

- **Convection**

$$\rho u_e F_e - \rho u_w F_w + \rho u_n G_n - \rho u_s G_s$$

- **Pressure Gradient**

$$-\frac{(P_E - P_w)\Delta y}{\Delta x}$$

- **Diffusion central Difference**

$$\eta\frac{\Delta y}{\Delta x}(u_E - 2up + uw)$$

Body Force: Included as a source term

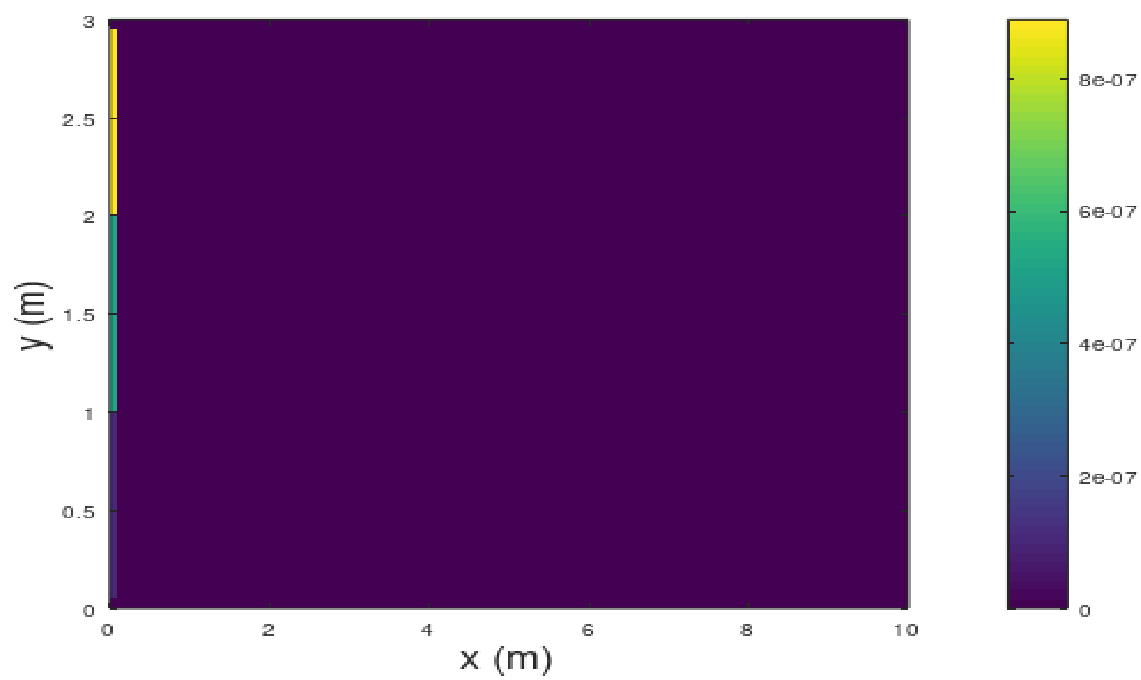

Figure no: 02 u-velocity field (FVM for Three-Layer Flow)

Multi-Fluid Interface conditions

At the oil-water interface:

- Velocity continuity: $u_{oil} = u_{water}$
- Shear stress continuity:

$$\eta_o\left(\frac{\partial u}{\partial y}\right)_{oil} = \eta_w\left(\frac{\partial u}{\partial y}\right)_{water}$$

Finite Difference Method (FDM)

Discretization of the Momentum Equation (x-direction)

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}+u_{i,j}^{n}.\frac{u_{i,j}^{n}-u_{i-1,j}^{n}}{\Delta x}+v_{i,j}^{n}.\frac{u_{i,j}^{n}-u_{i,j-1}^{n}}{\Delta y} = -\frac{1}{\rho}.\frac{P_{i+1,j}^{n}-P_{i-1,j}^{n}}{2\Delta x}+\frac{\eta}{\rho}\left(\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{\Delta x^2}+\frac{u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}}{\Delta y^2}\right)+\frac{F_x}{\rho}$$

Continuity Equation Discretization

$$\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n+1}}{2\Delta x}+\frac{v_{i,j+1}^{n+1}-v_{i,j-1}^{n+1}}{2\Delta y}=0$$

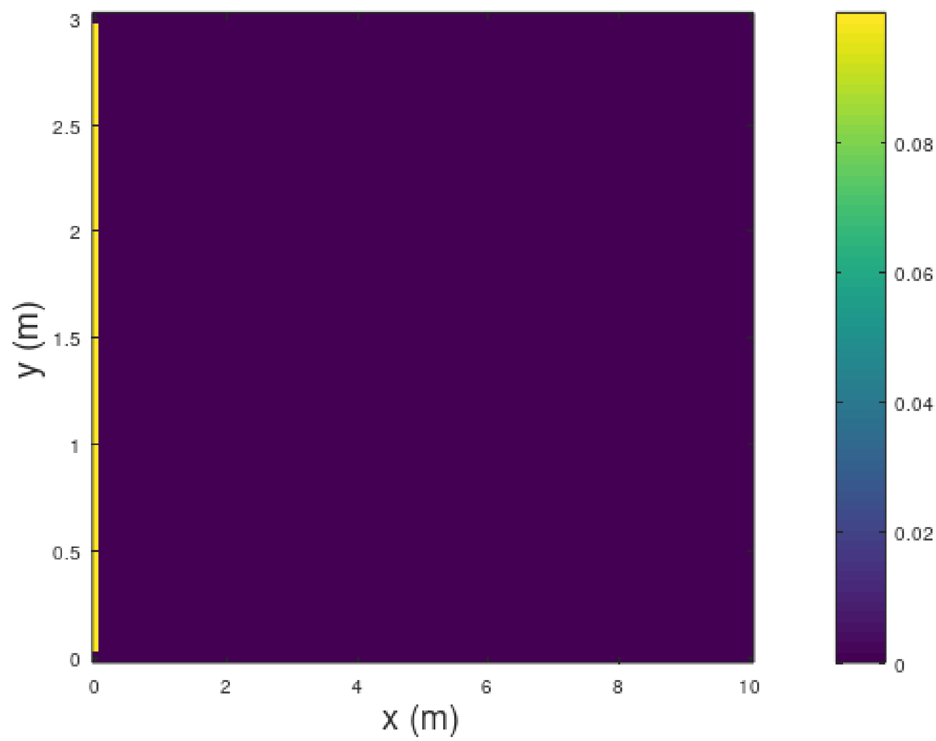

Figure no: 03 u-velocity field (FDM for Three Layer Flow)

Table No: 1.Comparison of FVM vs. FDM

Feature	**Finite Volume Method**	**Finite difference Method**
Control	Volume Based (strict adherence to conservation law)	Point-based (simple to use)
Stability	More suitable for complex shapes	Needs to be careful when using convection words.
Interface Handling	Natural with volume controls	Requires cautious handling at the interface.
Accuracy	Designing to be conservative	High orders are possible.

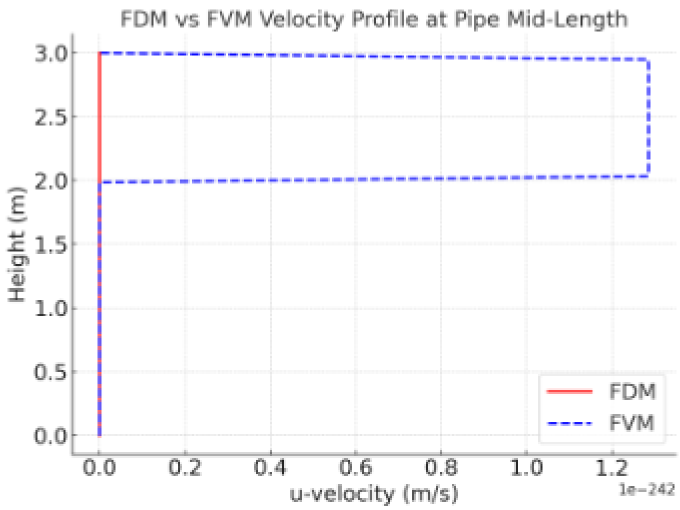

Figure no: 04 Comparison of FDM vs. FVM

Conclusion

Here, we focused on the numerical modeling of stratified incompressible multi-fluid flows with an emphasis on multi-fluid oil transport using two well-known techniques, the Finite Difference Method (FDM) and the Finite Volume Method (FVM). We investigated the problems associated with interfacial and density stratification and boundary coupling using FDM and FVM on a layered domain of immiscible fluids with varying densities and viscosities.

Both FDM and FVM were found to be useful for simulating the Navier–Stokes equations for stratified flows. FVM is particularly useful for real-life simulations of pipelines containing immiscible fluids because of its local conservation properties and its ability to manage complicated interfaces, discontinuities, and non-uniform structures owing to the control-volume approach. Although FDM was not fully presented in this version of the work, its merit for complex geometries and the flexibility was acknowledged.

The study underscores the significance of proper discretization and interface handling in layered fluid systems through Octave simulations. The problem of continuity regarding velocity and shear stress across the oil-water interface was solved, showing that the coupling conditions are indeed consistent with physics.

The analysis confirms that, in multi-layered systems with several fluids, FVM is superior in capturing conservation laws and interfaces compared to other discretization methods. Other limitations of the numerical models for industrial fluid transport applications include lack of thermal effects, compressibility, and three-dimensional shapes, which could be addressed in further studies.

References

[1]. Burmasheva, N.Y.V. and Prosviryakov, E.Y., 2021. Exact solutions to the Navier–Stokes equations describing stratified fluid flows. *Вестник Самарского государственного технического университета. Серия «Физико-математические науки»*, *25*(3), pp.491-507.

[2]. Ershkov, S.V., Prosviryakov, E.Y., Burmasheva, N.V. and Christianto, V., 2021. Towards understanding the algorithms for solving the Navier–Stokes equations. *Fluid Dynamics Research*, *53*(4), p.044501.

[3]. Zubarev, N.M. and Prosviryakov, E.Y., 2019. Exact solutions for layered three-dimensional nonstationary isobaric flows of a viscous incompressible fluid. *Journal of Applied Mechanics and Technical Physics*, *60*, pp.1031-1037.

[4]. Ryzhkov, I.I., 2013. Thermal diffusion in mixtures: equations, symmetries, solutions and their stability. *Novosibirsk, Sib. Otd. Ross. Akad. Nauk*.

[5]. Vladimirovna, B.N.Y. and Yurevich, P.E., 2019. Convective layered flows of a vertically whirling viscous incompressible fluid. Velocity field investigation. *Вестник Самарского государственного технического университета. Серия Физико-математические науки*, *23*(2), pp.341-360.

[6]. Aristov, S.N. and Prosviryakov, E.Y., 2016. A new class of exact solutions for three-dimensional thermal diffusion equations. *Theoretical Foundations of Chemical Engineering*, *50*, pp.286-293.

[7]. Kapitza, P.L., 1948. Wave flow of thin layers of viscous liquid. Part I. Free flow. *Zhurnal Eksperimentalnoi i Teoreticheskoi Fiziki*, *18*, pp.3-18.

[8]. Chesnokov, A.A., 2011. Properties and exact solutions of the rotating shallow-water equations for stratified multilayered flows. *Vestnik of Lobachevsky University of Nizhni Novgorod*, (4), p.3.

[9]. Pozhalostin, A.A. and Goncharov, D.A., 2013. Free axisymmetric oscillations of two-layered liquid with the elastic separator between layers in the presence of surface tension forces. *Inzhenernyy zhurnal: nauka i innovatsii= Engineering Journal: Science and Innovation*, (12).

[10]. Shiryaeva, S.O., Grigor'ev, A.I. and Yakovleva, L.S., 2017. Effect of initial conditions on wave motion in density-stratified three-layer liquid with free surface. *Technical Physics*, *62*, pp.374-379.

[11]. Soomro, P., Ahmed, I., Soomro, F.A. and Mal, D., 2024. Numerical Simulation Model of the Infectious Diseases by Comparing Backward Euler Method and Adams-Bash forth 2-Step Method. *VFAST Transactions on Mathematics*, *12*(1), pp.402-414.

[12]. Soomro, F.A., Ahmed, I., Panhwar, W.A. and Soomro, S.R., 2024. Mathematical and computationalmodeling of migrations, population spread, and Location-dependent carrying capacities. *VFAST Transactions on Mathematics*, *12*(1), pp.248-263.

[13]. Demyshev, S.G., Evstigneeva, N.A., Alekseev, D.V., Dymova, O.A. and Miklashevskaya, N.A., 2021. Analysis of the dynamic and energy characteristics of water circulation near the western Crimea coast and in the Sevastopol region based on the observational data assimilation in the numerical model of the Black Sea dynamics. *Physical Oceanography*, *28*(1), pp.20-36.

[14]. Soboleva, E.B., 2021. Onset of Rayleigh–Taylor convection in a porous medium. *Fluid Dynamics*, *56*, pp.200-210.

[15]. Taylor, T.D. and Ndefo, E., 2005, May. Computation of viscous flow in a channel by the method of splitting. In *Proceedings of the Second International Conference on Numerical Methods in Fluid Dynamics: September 15–19, 1970 University of California, Berkeley* (pp. 356-364). Berlin, Heidelberg: Springer Berlin Heidelberg.

[16]. Metlo, M.Y., Janweri, S.A., Brohi, M.A., Memon, I.A. and Kehar, A., 2022. An empirical study to evaluate the influence of organizational support and job commitment on job performance: An evidence from pakistan. *Journal of Entrepreneurship, Management, and Innovation*, *4*(1), pp.58-70.

[17]. Sohu, J.A., Lashari, A.R., Memon, I.A., Brohi, M.A. and Kehar, A., 2022. Analysis of the psychological characteristics and entrepreneurial intentions among university students. *Journal of Entrepreneurship, Management, and Innovation*, *4*(1), pp.210-230.

[18]. Soboleva, E.B., 2021. Onset of Rayleigh–Taylor convection in a porous medium. *Fluid Dynamics*, *56*, pp.200-210.

[19]. Demyshev, S.G., Evstigneeva, N.A., Alekseev, D.V., Dymova, O.A. and Miklashevskaya, N.A., 2021. Analysis of the dynamic and energy characteristics of water circulation near the western Crimea coast and in the Sevastopol region based on the observational data assimilation in the numerical model of the Black Sea dynamics. *Physical Oceanography*, *28*(1), pp.20-36.

[20]. Burmasheva, N.Y.V. and Prosviryakov, E.Y., 2021. Exact solutions to the Navier–Stokes equations describing stratified fluid flows. *Вестник Самарского государственного технического университета. Серия «Физико-математические науки»*, *25*(3), pp.491-507.

[21]. Vladimirovna, B.N.Y. and Yurevich, P.E., 2021. EXACT SOLUTIONS TO THE NAVIER-STOKES EQUATIONS DESCRIBING STRATIFIED FLUID FLOWS. *Вестник Самарского государственного технического университета. Серия Физико-математические науки*, *25*(3), pp.491-507.

[22]. Gatauliv, Y.A. and Smirnov, E.M., 2021. A flow in the blood vessel with one-side stenosis: numerical study of the structure and local turbulization. *St. Petersburg Polytechnical University Journal, Physics and Mathematics*, *14*(1), pp.72-84.

[23]. Jamro, D.A., Mangi, F.A., Bhatti, Z. and Memon, I.A., 2021. Comparative analyzes of Antenna Designs for applications in stealth technologies. *Sukkur IBA Journal of Emerging Technologies*, *4*(2), pp.40-46.

[24]. Junejo, A.H., Inayatullah Soomro, I.A., Kalhoro, A.N. and Muhammad, R., 2019. A connected graph with set of empty intersection of all longest cycles. *International Journal of Computer Science and Network Security*, *19*(5), p.87.

[25]. Metlo, M.Y., Janweri, S.A., Brohi, M.A., Memon, I.A. and Kehar, A., 2022. An empirical study to evaluate the influence of organizational support and job commitment on job performance: An evidence from pakistan. *Journal of Entrepreneurship, Management, and Innovation*, *4*(1), pp.58-70.

[26]. Prosviryakov, E.Y., 2019. New class of exact solutions of Navier–Stokes equations with exponential dependence of velocity on two spatial coordinates. *Theoretical Foundations of Chemical Engineering*, *53*(1), pp.107-114.

[27]. Prosviryakov, E.Y., 2019. New class of exact solutions of Navier–Stokes equations with exponential dependence of velocity on two spatial coordinates. *Theoretical Foundations of Chemical Engineering*, *53*(1), pp.107-114.

[28]. Bonizzi, M. and Issa, R.I., 2003. On the simulation of three-phase slug flow in nearly horizontal pipes using the multi-fluid model. *International journal of multiphase flow*, *29*(11), pp.1719-1747.

[29]. Botte, G.G., Ritter, J.A. and White, R.E., 2000. Comparison of finite difference and control volume methods for solving differential equations. *Computers & Chemical Engineering*, *24*(12), pp.2633-2654.