Online ISSN 3007-3197
Print ISSN 3007-3189
http://amresearchreview.com/index.php/Journal/about

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 3, Issue 6 (2025)

Maximal Ideals in Near-Ring

[1]Ambreen Zehra, [2]Nazra Zahid Shaikh, [3]Prof. Dr. Sarwar Jahan Abbasi

Article Details

Keywords: Commutative Near-ring, near-ring with Identity, Maximal Ideals, Near Field, Prime Ideals, Quotient Near ring.

Ambreen Zehra
ambreen.zehra@hamdard.edu.pk
Nazra Zahid Shaikh
nazra.zahid@hamdard.edu.pk
Prof. Dr. Sarwar Jahan Abbasi
sarwarjabbasi@yahoo.com

ABSTRACT

We have been studied near-ring with identity and commutative near-ring. We show that:
i)Every proper ideal is contained in a maximal ideal if R is near-ring with identity.
ii)R is a near field if and only if 0 is a maximal ideal.
iii)M is a maximal ideal of R if and only if the quotient near-ring $\frac{R}{M}$ is near field.
iv)The ideal P is a prime ideal of R if and only if $\frac{R}{P}$ is near integral domain.
v)Every maximal ideal of R is prime ideal.

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 3, Issue 6 (2025)

THE COMMENCEMENTS AND DEVELOPMENT OF NEAR-RING THEORY

In 1905, Near-ring were revealed as a near-field by Leonard E. Dickson. He constructed an algebraic structure that had all the properties of a field excluding one distributive law missing. The generalization of ring with one distributive law and addition is not necessarily commutative is near-ring. Fifty-six years ago, late in 1968, the first conference on near-ring and near-fields took place in the history of mathematics in Germany. After that mathematicians have found much application in area like cryptography, group theory, geometry and its other branches and coding theory. The work on near-ring is still in progress since its applications are growing.

SOME BASIC DEFINITIONS AND RESULTS

We recall necessary definitions of near ring, Near ring with identity, commutative near ring, Maximal ideals, Quotient near ring, Near field and Prime ideal.

BASIC DEFINITIONS

A set R with two binary operations (+) and (.) is right near ring if (R, +) is a group, (R,.) is a semi group and (x+ y) z= x z +y z for all x, y and z in R.

If (R*(=R\ {0}),.) is also a group then R is called a near field.

$u\epsilon R$ is called a unit if there exists $v\epsilon R$ such that u v =1.

A near ring (R, +,) is called commutative if (R,.) is a commutative semi group. Near ring without zero divisors is called near integral domain.

$P \lhd R$ is called prime if for all $I, J \lhd R: IJ \subseteq P => I \subseteq P \quad J \subseteq P$ 1 .

A Maximal ideal of R is an ideal which is maximal in the set of all non-zero ideals.

Let R be a near ring I an ideal of R. Let R/I = $\{I + r; r \in R\}$ be the set of all additive cosets of I in R. Then (R/I, +,.) is called the quotient near ring of R by or over I, where + and . are defined by (I, + r_1) +(I, r_2) = I + r_1 + r_2 and (I, +r_1) .(I, r_2) = I +r_1r_2 for all r_1 ,r_2 in R [2].

$I, +$ is a normal subgroup of $R, +$, we have $R + i = i + R \forall i \in I$

So we could equally well have

$i + r \ s - rs \in I \forall i \in I and r, s \in R$

Let $R, +, .$ be a near ring. An ideal of R is a subset I of R such that

i. $I, +$ is a normal subgroup of $R, +$.
ii. $RI \subseteq I,$
iii. $r + i \ s - rs \in I r, s \in R.$

If I satisfies (i) and (ii) then it is called a left ideal of$R$. If I satisfies (i) and (iii) then it is called a right ideal of R [5].

PRE-REQUISITE RESULT

PROPOSITION

For all $a, b \in R$, (i) 0a = 0 and (ii) (-a) b = -a b [4].

LATTICE ISOMORPHIC THEOREM

Let R be a left near ring and M be an ideal of R. The correspondence between A and A/M is an inclusion preserving bijection between the set of sub near ring A of R that contain M and the set of sub near ring of A/M. Furthermore, A is a sub near ring containing M, is an ideal of R if and only if A/M is an ideal of R/M [8].

ZORN'S LEMMA

A partially ordered non-empty set in which every chain is bounded above respectively below has a maximal and minimal element.

A non-empty set X with a partial order ≤ is called a poset, i.e., ≤ satisfies the following:
Reflexivity: x $x \forall x \in X$

Anti-Symmetry: $x \leq y and y \leq x \rightarrow x = y$

Transitivity: $x \leq y and y \leq z \rightarrow x \leq z$

A subset Y and X is called a chain or said to be totally ordered if any two elements of y are comparable, i.e., given $x, y \in Y$, either$x \leq y and y \leq x$.

In particular, given finitely many elements $y_1 \ldots \ldots \ldots \ldots y_n$ in Y, there is a permutation of 1…..n. Such that $y_\sigma \; 1 \; \leq \cdots \ldots \ldots \leq y_\sigma \; n$. A subset A of X is said to be bounded above and below if there is an $a \in X$ such that $\leq a \forall x \in A$. Such an ―a‖ is called an upper and respectively lower bound for A in X. It need not belong to A.

A subset A of X is said to have a maximal resp. minimal element if there is an $a \in A$ such that
$a < x \; resp. x < a \; \forall x \in A, x \neq a.$

Note that a maximal resp. minimal element need not exist or need not be an upper resp. a lower bound when exists or it need not be unique.

Zorn's Lemma guarantees the existence of maximal left/right/2-sided ideals in a ring R with unity [8].

OUR MAIN WORK

Now, here we prove interesting results related to maximal ideals and prime ideals in near ring.

THEOREM

Every proper ideal is contained in a maximal ideal if R is near ring with identity.

PROOF

Let R be the near ring with identity ($1 \neq 0$) and I be the proper ideal R. R cannot be the zero near ring. Let R be the set of all proper ideals of R which contain I, then S is non-empty because I

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 3, Issue 6 (2025)

belongs to S. Let C be the chain in S of proper ideals define on set J to be the union of ideals in C.

$$J = U_{A\epsilon C}A$$

Now,

We have to show that

I. J is an ideal.
II. J is proper left ideal.

We prove J is an ideal then we have to show that

I. $(J, +)$ is a normal subgroup of $(R, +)$.
II. $RI \subseteq I$,
III. $r + j\ s - rs \in J as \forall j \in J and r, s \in R$

(j, +) Is a Normal Subgroup Of (r, +)

For a normal subgroup we need to show that

$$R + J = J + R$$

i.e., both cosets coincide.

Now

r ∈ R and j ∈J. We have to show that

$$r + j = j + r$$

$\Longrightarrow r - r + j = j - r + r$ Adding ―$-r$‖ on both sides for –r belongs to R

$\Longrightarrow 0 + j = j + 0$

$\Longrightarrow j = j$ Addition by identity

Which shows that (, +) is a normal subgroup of (R, +).

_ ⊆

We need to show that r j ∈J.

For r ∈R and j ∈J which is obvious as J is a collection of ideals and each is closed under multiplication.

(r + j)s –r s ∈J ∀j ∈J and r, s ∈R We have

$\Longrightarrow rj \in J$ by above

$\Longrightarrow rj + rs - rs = rj$ Add and subtract rs on left side

$\Longrightarrow r(j + s) - rs = rj$ $rj \in J \& rs \in R$

Hence J is an ideal.

J IS PROPER LEFT IDEAL

We start proof with contradiction. Lets J is not proper ideal

Then 1 ∈J by ideals properties

1 ∈A for some A ∈C

Which is contradiction because each A is a proper ideal? Hence J is a proper ideal by Zorn's Lemma [2.2.3].

If A is a non-empty partially ordered set in which every chain has an upper bound then A has a MAXIMAL ELEMENT. S HAS A MAXIMAL WHICH IS THEREFORE A MAXIMAL IDEAL CONTAINING I.

THEOREM

R is a near field if and only if 0 is a maximal ideal.

PROOF

Assume R is near field and I is an ideal in R. Such that I ≠ 0

Then I contain a non-zero element, say ―a ‖

$a^{-1} \in R$ Since R is near field

$\Rightarrow aa^{-1} \in I$ due to absorbing property in I

$\Rightarrow 1 \in I$ Since I is the multiplicative identity

$\Rightarrow 1.r \in I$ for all r $\in R$

$\Rightarrow r \in I$ Since r = 1.r

$\Rightarrow I = R$.

Hence, 0 is a maximal ideal. Conversely,

$I = R$

$\Rightarrow 1 \in R$ Given

$\Rightarrow aa^{-1} \in R$ by definition of multiplicative identity

$\Rightarrow a^{-1} \in R$ by closure law with respect to multiplication in R

Hence, R is near-field by definition [2.2].

THEOREM

M is a maximal ideal of R if and only if the quotient near ring $\frac{R}{M}$ is near field.

PROOF

Since $M \leq I \leq R$ by 2.2.2

So, if R is commutative near ring (2.1), then R is near field (2.1) if and only if its only ideals are 0 and R. Hence $\frac{R}{M}$ is a near field.Conversely, Suppose $\frac{R}{M}$ is near field(1.1) $\frac{R}{M} \neq \oint$. Let M≠ R. I is an ideal. Such that $M \neq I, M \leq I \leq R$ then if a ∈ I.

R

$\Rightarrow$ a + M $\neq$ 0 in —

M

R

$\Rightarrow$ a + M is invertiable in —

M

$\Rightarrow$ a + M b + M = ab + M = 1 + M

$\Rightarrow$ ab − 1 $\in$ M $\leq$ I

$\Rightarrow$ ab − 1 $\in$ I

Since1 $\in$ I, whenever $a - b \in$ I

$\Rightarrow$ I = M for $a \in$ I, $b \in$ I.

Hence M is a maximal ideal[9].

THEOREM

The ideal P is a prime ideal of R if and only if R is near integral domain.

P

PROOF

Let P is a prime ideal in R.

$P \neq R$ i.e., $a \notin P$ or $b \notin P$

either $a \in P$ or $b \in P$

Then if $a \in P$

Let $a + P \in R/P$ and $b + P \in R/P$. Then $a + P$ $b + P$ $= I$

Hence, R/P is an integral domain [2.1].

Conversely, P is an ideal of R. R/P is an integral domain. Let $a, b \in R$.

Such that $ab \in P$.

Then either $a \in P$ or $b \in P$

$=$ $a + P$ $b + P$ by coset multiplication

$= ab + P$ therefore $ab \in P$

$= P$

i.e., $a + P$ $b + P$ $= 0$ in R/P. then R/P is an integral domain.

Therefore, we have $a + P$ $= P$ or $b + P$ $= P$

i.e., $a \in P$ or $b \in P$

Hence P is a prime ideal by definition of prime ideal [2.1][9].

THEOREM

Every maximal ideal of R is prime ideal.

PROOF

Let M be a maximal ideal of commutative near ring with unity.

$\frac{R}{M}$ is near field (Theorem 3.2) and every near field is an integral domain.

Then $\frac{R}{M}$ is an integral domain (Theorem 3.3).

Hence, every maximal ideal is a prime ideal.

REFERENCES

[1] Pilz, G. (2011). *Near-rings: the theory and its applications.* Elsevier.

[2] Meldrum, J. D. P., & Meyer, J. H. (1996). Intermediate ideals in matrix near- rings. *Communications in algebra*, *24*(5), 1601-1619.

[3] Allenby, R.B.J.(1983). Rings, Field and Groups,Edward Arnold (Publishers) Ltd, 104-106.

[4] Pilz, G. (1983), Near rings, Second Edition, North Holland/ American Elsevier, Amsterdam.

[5] Meldrum, J. D. P , (1985). Near ring and their links with groups, Research notes in Mathematics, Pitman.

[6] Scott, W.R. (1964). Group theory, Prentice Hall, Englewood Cliffs. N. J.

[7] David S. Dummit, Foote Richhard , M. (2002), Abstract Algebra, Second Edition, John wiley and Sons.

[8] C. Musili, 1992, Introduction to Rings and Modules, Narosa Publishing House for Toppan Company (S) Pte Ltd, Singapore, ISBN 981-00—3181-5.

[9] S.J. Abbasi and A. Zehra, ―Study of Prime ideals in Near-Ring‖ , Technol. forces, ISSN 1994- 862x, Vol.02, pp.65-66, 2008.